Lecture 16

Waves in Layered Media

16.1 Waves in Layered Media

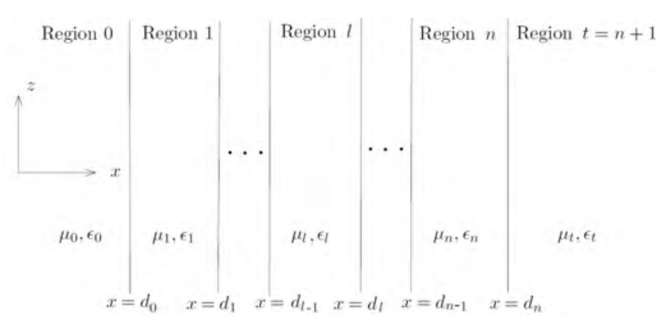

Figure 16.1: Figure for layered media borrowed from Kong's book. Please note that in our notes, the first region is Region 1. We shall also, replace x with z and vice versa (courtesy of J.A. Kong, Electromagnetic Wave Theory).

Because of the homomorphism between the transmission line problem and the plane-wave reflection by interfaces, we will exploit the simplicity of the transmission line theory to arrive at formulas for plane wave reflection by layered media. This treatment is not found in any other textbooks.

16.1.1 Generalized Reflection Coefficient for Layered Media

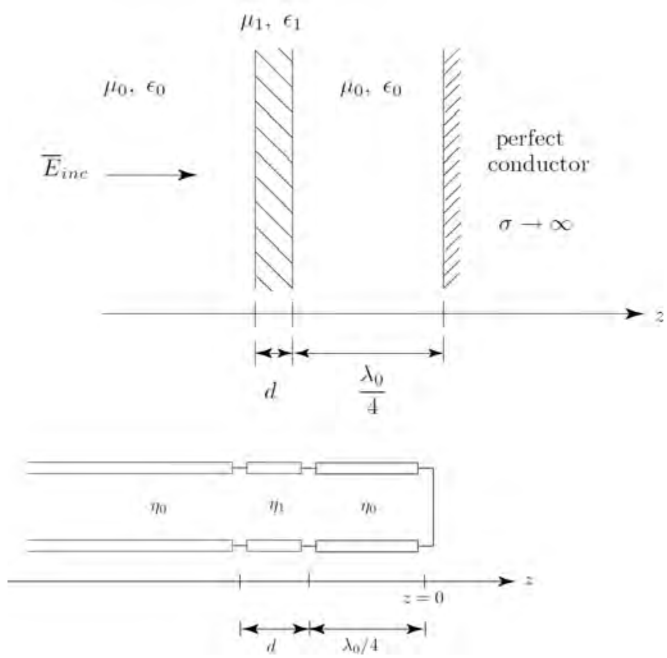

Figure 16.2: The equivalence of a layered medium problem to a transmission line problem. This equivalence is possible even for oblique incidence. For normal incidence, the wave impedance becomes intrinsic impedances (courtesy of J.A. Kong, Electromagnetic Wave Theory).

Because of the homomorphism between transmission line problems and plane waves in layered medium problems, one can capitalize on using the multi-section transmission line formulas for generalized reflection coefficient, which is

$$\tilde{\Gamma}_{12} = \frac{\Gamma_{12} + \tilde{\Gamma}_{23} e^{-2j\beta_2 l_2}}{1 + \Gamma_{12}\tilde{\Gamma}_{23} e^{-2j\beta_2 l_2}} \tag{16.1.1}$$

This reflection coefficient includes multiple reflections from the right of the 1,2 junction. It can be used to study electromagnetic waves in layered media shown in Figures 16.1 and 16.2.

Using the result from the multi-junction transmission line, we can write down the generalized reflection coefficient for a layered medium with an incident wave at the 1,2 interface, including multiple reflections from the right. It is given by

$$\tilde{R}_{12} = \frac{R_{12} + \tilde{R}_{23} e^{-2j\beta_{2z} l_2}}{1 + R_{12}\tilde{R}_{23} e^{-2j\beta_{2z} 2l_2}} \tag{16.1.2}$$

where l_2 is now the thickness of the region 2. In the above, we assume that the wave is incident from medium 1 which is semi-infinite, the generalized reflection coefficient above is defined at the media 1 and 2 interface.[1] It is assumed that there are multiple reflections coming from the right of the 2,3 interface, so that the 2,3 reflection coefficient is the generalized reflection coefficient $\tilde{R}_{23}$.

Figure 16.2 shows the case of a normally incident wave into a layered media. For this case, the wave impedance becomes the intrinsic impedance of homogeneous space.

We shall discuss finding guided waves in a layered medium next using the generalized reflection coefficient. For a general guided wave along the longitudinal direction parallel to the interfaces (x direction in our notation), the wave will propagate in the manner of

$$e^{-j\beta_x x}$$

For instance, the surface plasmon mode that we found previously can be thought of as a wave propagating in the x direction. This wave has very interesting phase and group velocities. Hence, it is prudent to understand what phase and group velocities are before doing this.

16.2 Phase Velocity and Group Velocity

Now that we know how a medium can be frequency dispersive in the Drude-Lorentz-Sommerfeld (DLS) model, we are ready to distinguish the difference between the phase velocity and the group velocity

16.2.1 Phase Velocity

The phase velocity is the velocity of the phase of a wave. It is only defined for a monochromatic signal (also called time-harmonic, CW (constant wave), or sinusoidal signal) at one given frequency. A sinusoidal wave signal, e.g., the voltage signal on a transmission line in the time domain, can take the form

$$\begin{aligned} V(z,t) &= V_0 \cos(\omega t - kz + \alpha) \\ &= V_0 \cos\left[k\left(\frac{\omega}{k}t - z\right) + \alpha\right] \end{aligned} \tag{16.2.1}$$

This sinusoidal signal moves with a velocity

$$v_{ph} = \frac{\omega}{k} \tag{16.2.2}$$

where, for example, $k = \omega\sqrt{\mu\varepsilon}$, inside a simple coax. Hence,

$$v_{ph} = 1/\sqrt{\mu\varepsilon} \tag{16.2.3}$$

But a dielectric medium can be frequency dispersive, or $\varepsilon(\omega)$ is not a constant but a function of ω as has been shown with the Drude-Lorentz-Sommerfeld model. Therefore, signals with different ω's will travel with different phase velocities.

[1] We have borrowed Figure 16.1 from Kong's book, where the first region is Region 0. But in our lecture, the first region is Region 1.

More bizarre still, what if the coax is filled with a plasma medium where

$$\varepsilon = \varepsilon_0 \left(1 - \frac{\omega_p{}^2}{\omega^2}\right) \tag{16.2.4}$$

Then, $\varepsilon < \varepsilon_0$ always meaning that the phase velocity given by (16.2.3) can be larger than the velocity of light in vacuum (assuming $\mu = \mu_0$). Also, $\varepsilon = 0$ when $\omega = \omega_p$, implying that $k = 0$; then in accordance to (16.2.2), $v_{ph} = \infty$. These ludicrous observations can be justified or understood only if we can show that information can only be sent by using a wave packet.[2] The same goes for energy which can only be sent by wave packets, but not by CW signal; only in this manner can a finite amount of energy be sent. These wave packets can only travel at the group velocity as shall be shown, which is always less than the velocity of light.

16.2.2 Group Velocity

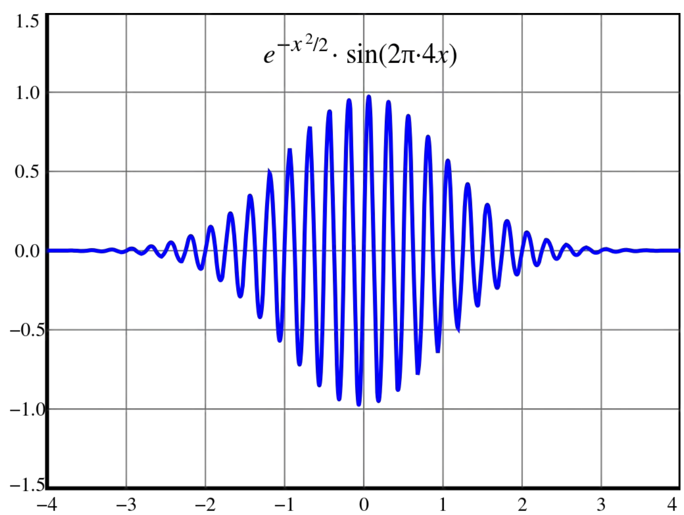

Figure 16.3: A Gaussian wave packet can be thought of as a linear superposition of monochromatic waves of slightly different frequencies. If one Fourier transforms the above signal, it will be a narrow-band signal centered about certain ω_0 (courtesy of Wikimedia [101]).

Now, consider a narrow band wave packet as shown in Figure 16.3. It cannot be monochromatic, but can be written as a linear superposition of many frequencies. One way to express this is to write this wave packet as an integral in terms of Fourier transform, or a

[2]In information theory, according to Shannon, the basic unit of information is a bit, which can only be sent by a digital signal, or a wave packet.

summation over many frequencies, namely

$$V(z,t) = \int_{-\infty}^{\infty} d\omega V(z,\omega) e^{j\omega t} \tag{16.2.5}$$

Assume that $V(z,t)$ is the solution to the dispersive transmission line equations with $\varepsilon(\omega)$, then it can be shown that $V(z,\omega)$ is the solution to the one-dimensional Helmholtz equation[3]

$$\frac{d^2}{dz^2} V(z,\omega) + k^2(\omega) V(z,\omega) = 0 \tag{16.2.6}$$

When the dispersive transmission line is filled with dispersive material, then $k^2 = \omega^2 \mu_0 \varepsilon(\omega)$. Thus, upon solving the above equation, one obtains that $V(z,\omega) = V_0(\omega) e^{-jkz}$, and

$$V(z,t) = \int_{-\infty}^{\infty} d\omega V_0(\omega) e^{j(\omega t - kz)} \tag{16.2.7}$$

In the general case, k is a complicated function of ω as shown in Figure 16.4.

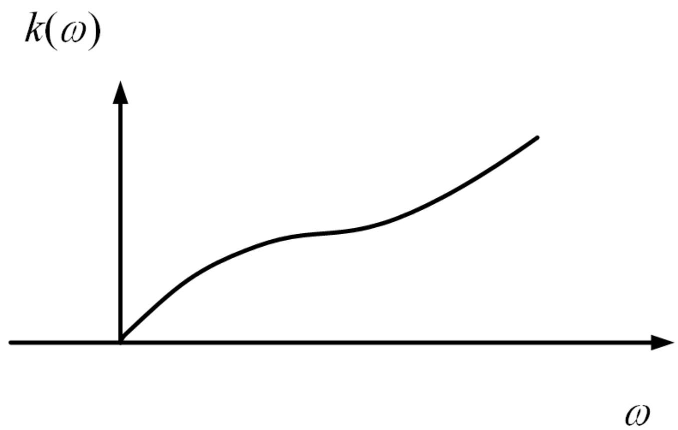

Figure 16.4: A typical frequency dependent $k(\omega)$ albeit the frequency dependence can be more complicated than shown.

Since this is a wave packet, we assume that $V_0(\omega)$ is narrow band centered about a frequency ω_0, the carrier frequency as shown in Figure 16.5. Therefore, when the integral in (16.2.7) is performed, it needs only be summed over a narrow range of frequencies in the vicinity of ω_0.

[3]In this notes, we will use k and β interchangeably for wavenumber. The transmission line community tends to use β while the optics community uses k.

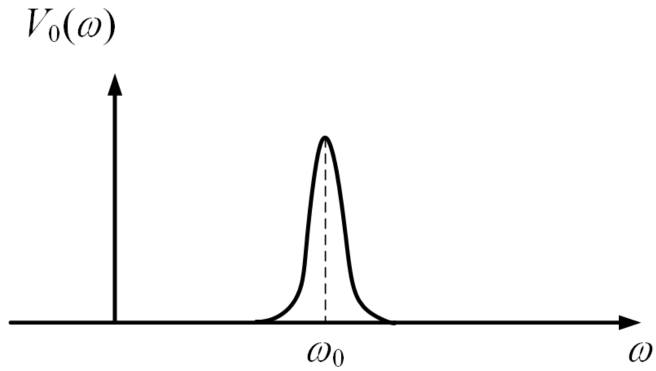

Figure 16.5: The frequency spectrum of $V_0(\omega)$.

Thus, we can approximate the integrand in the vicinity of $\omega = \omega_0$, and let

$$k(\omega) \cong k(\omega_0) + (\omega - \omega_0)\frac{dk(\omega_0)}{d\omega} + \frac{1}{2}(\omega - \omega_0)^2 \frac{d^2k(\omega_0)}{d\omega^2} + \cdots \tag{16.2.8}$$

To ensure the real-valuedness of (16.2.5), one ensures that $-\omega$ part of the integrand is exactly the complex conjugate of the $+\omega$ part. Another way is to sum over only the $+\omega$ part of the integral and take twice the real part of the integral. So, for simplicity, we rewrite (16.2.5) as

$$V(z,t) = 2\Re e \int_0^\infty d\omega V_0(\omega) e^{j(\omega t - kz)} \tag{16.2.9}$$

Since we need to integrate over $\omega \approx \omega_0$, we can substitute (16.2.8) into (16.2.9) and rewrite it as

$$V(z,t) \cong 2\Re e \left[e^{j[\omega_0 t - k(\omega_0)z]} \underbrace{\int_0^\infty d\omega V_0(\omega) e^{j(\omega-\omega_0)t} e^{-j(\omega-\omega_0)\frac{dk}{d\omega}z}}_{F\left(t - \frac{dk}{d\omega}z\right)} \right] \tag{16.2.10}$$

where more specifically,

$$F\left(t - \frac{dk}{d\omega}z\right) = \int_0^\infty d\omega V_0(\omega) e^{j(\omega-\omega_0)t} e^{-j(\omega-\omega_0)\frac{dk}{d\omega}z} \tag{16.2.11}$$

It can be seen that the above integral now involves the integral summation over a small range of ω in the vicinity of ω_0. By a change of variable by letting $\Omega = \omega - \omega_0$, it becomes

$$F\left(t - \frac{dk}{d\omega}z\right) = \int_{-\Delta}^{+\Delta} d\Omega V_0(\Omega + \omega_0) e^{j\Omega\left(t - \frac{dk}{d\omega}z\right)} \tag{16.2.12}$$

When Ω ranges from $-\Delta$ to $+\Delta$ in the above integral, the value of ω ranges from $\omega_0 - \Delta$ to $\omega_0 + \Delta$. It is assumed that outside this range of ω, $V_0(\omega)$ is sufficiently small so that its value can be ignored.

The above itself is a Fourier transform integral that involves only the low frequencies of the Fourier spectrum where $e^{j\Omega\left(t - \frac{dk}{d\omega}z\right)}$ is evaluated over small Ω values. Hence, F is a slowly varying function. Moreover, this function F moves with a velocity

$$v_g = \frac{d\omega}{dk} \tag{16.2.13}$$

Here, $F(t - \frac{z}{v_g})$ in fact is the velocity of the envelope in Figure 16.3. In (16.2.10), the envelope function $F(t - \frac{z}{v_g})$ is multiplied by the rapidly varying function

$$e^{j[\omega_0 t - k(\omega_0)z]} \tag{16.2.14}$$

before one takes the real part of the entire function. Hence, this rapidly varying part represents the rapidly varying carrier frequency shown in Figure 16.3. More importantly, this carrier, the rapidly varying part of the signal, moves with the velocity

$$v_{ph} = \frac{\omega_0}{k(\omega_0)} \tag{16.2.15}$$

which is the phase velocity.

16.3 Wave Guidance in a Layered Media

Now that we have understood phase and group velocity, we are at ease with studying the propagation of a guided wave in a layered medium. We have seen that in the case of a surface plasmonic resonance, the wave is guided by an interface because the Fresnel reflection coefficient becomes infinite. This physically means that a reflected wave exists even if an incident wave is absent or vanishingly small. This condition can be used to find a guided mode in a layered medium, namely, to find the condition under which the generalized reflection coefficient (16.1.2) will become infinite.

16.3.1 Transverse Resonance Condition

Therefore, to have a guided mode exist in a layered medium, the denominator of (16.1.2) is zero, or that

$$1 + R_{12}\tilde{R}_{23}e^{-2j\beta_{2z}l_2} = 0 \tag{16.3.1}$$

where t is the thickness of the dielectric slab. Since $R_{12} = -R_{21}$, the above can be written as

$$1 = R_{21}\tilde{R}_{23}e^{-2j\beta_{2z}l_2} \tag{16.3.2}$$

The above has the physical meaning that the wave, after going through two reflections at the two interfaces, 21, and 23 interfaces, which are R_{21} and R_{23}, plus a phase delay given by $e^{-2j\beta_{2z}l_2}$, becomes itself again. This is also known as the transverse resonance condition. When specialized to the case of a dielectric slab with two interfaces and three regions, the above becomes

$$1 = R_{21}R_{23}e^{-2j\beta_{2z}l_2} \tag{16.3.3}$$

The above can be generalized to finding the guided mode in a general layered medium. It can also be specialized to finding the guided mode of a dielectric slab.